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Analytic $\mathcal{O}(\alpha)$ Results for Bottom and Top Quark Production in e^+e^- Collisions

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ABSTRACT

We present a new derivation of the $\mathcal{O}(\alpha)$ angular distribution of the outgoing q-quark in the production process $e^+e^- \to \gamma, Z \to q\,\bar{q}(g)$. In our calculation, we express the three-particle phase-space integration of the gluon-bremsstrahlung process in terms of a general set of analytic integral solutions. A consistent treatment of the QCD one-loop corrections to the axial-vector current deserves special attention. This is relevant in the derivation of the forward-backward asymmetry predicted by the standard model. Finally, we provide the full analytical solutions for the differential rates in closed form and conclude with numerical estimates for bottom and top quark production.

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1 Introduction

The recent discovery of the top quark has intensified interest for radiative corrections in the Standard Model. In particular, processes involving top pair production are very attractive for closer investigation due to the absence of fragmentation effects. For bottom quark production, the measurement of the forward-backward asymmetry from Z-decays allows for a very precise determination of the electroweak mixing angle and the vector/axial-vector couplings to the fermions. The high accuracy of these experiments as carried out by the LEP [1] and SLC [2] collaborations requires theoretical predictions beyond Born level in the perturbative expansion of electromagnetic and strong couplings.

In this regard, the calculation of differential cross sections for the electroweak production of massive quarks from e^+e^- annihilation is of crucial importance. Radiative corrections to this process are in general dominated by QCD effects, which can be calculated perturbatively at energies sufficiently above the production threshold of the quark pair.

A first analytical treatment of radiative corrections to the angular distributions in heavy-quark production was carried out in Ref. [3] at energies where the e^+e^- annihilation process is dominated by pure photon exchange. At higher energies, the $\gamma - Z$ interference has to be included [4]. This leads to a considerable complication in the three-body phase-space (gluon bremsstrahlung off a massive quark pair). Only recently [5], expressions for the bremsstrahlung process have been obtained by including cuts on the energy of the emmitted gluon.

In this article, we present a systematic derivation of the QCD one-loop corrections to the differential production rate for the annihilation process $e^+e^- \to \gamma, Z \to q\,\bar{q}(g)$. No approximations are made and the whole three-body phase-space is considered. The full dependence on the quark masses is kept so that the final analytical expressions apply to the full energy spectrum where perturbative QCD is valid.

Our approach relies on a general set of integral solutions completely describing the complicated phase-space with two massive fermions and one vector particle. The method involves special integration techniques and was devised in Ref. [6]. It was used in various

calculations at $\mathcal{O}(\alpha)$ such as the total longitudinal or beam-alignment polarization for quarks produced in e^+e^- annihilation [7,8,9]. In the present work, we have been lead to an additional different class of phase-space integrals due to the more intricate angular structure of the differential calculation. Furthermore, we have used two different γ_5-prescriptions in the derivation of the $\mathcal{C}$-odd part of the differential cross section. There, the corresponding traces contain an odd number of the Dirac γ_5 and one faces immediately the problem on how to treat it consistently within dimensional regularization.

The article is organized as follows. In Section 2, we set up the basic formalism for the calculation of the differential cross section $d\sigma(e^+e^- \to \gamma, Z \to q\bar{q})/d\cos\theta$ (the angle θ is the usual scattering of the q-quark with respect to the electron beam axis) including Born approximation and QCD virtual corrections. Special attention is paid to the gluonic vertex corrections within different regularization schemes. In Section 3 follows a detailed presentation of the kinematics of the massive $q\bar{q}g$ phase-space which directly gives the angular distributions for the $\mathcal{O}(\alpha)$ tree-graph contributions. Compact expressions for the real-gluon corrections are derived. The $\mathcal{C}$-odd partial rate is calculated in two independent approaches yielding identical results. In Section 4, we present the full analytical results for the differential rates and conclude with an explicit numerical analysis for bottom and top quark production. The article is supplemented by two Appendices A and B where details on the renormalization of the axial-vector current and the massive three-body phase-space can be found.

2 Basic Framework and Virtual Corrections

The differential cross section for the production process $e^+e^- \to q\bar{q}$ is a binomial in $\cos\theta$, θ being the scattering angle of the tagged quark. It is common to introduce the structure functions σ_U, σ_L, and σ_F as follows

$$\frac{d\,\sigma}{d\cos\theta} = \tfrac{3}{8}(1+\cos^2\theta)\,\sigma_U + \tfrac{3}{4}\sin^2\theta\,\sigma_L + \tfrac{3}{4}\cos\theta\,\sigma_F, \tag{1}$$

so that σ_U and σ_L are the contributions stemming from unpolarized and longitudinally polarized gauge bosons, respectively. The structure function σ_F relates to the difference of left- and right-chiral polarizations of the quarks and constitutes the $\mathcal{C}$-odd component

under the exchange of quark and antiquark in the final state. All structure functions σ_i $(i = U, L, F)$ contain the electroweak couplings of the process under consideration.

For the calculation of the different terms in Eq. (1) (corresponding to Born and virtual corrections as shown in Fig. 1a,b) it is more convenient to rewrite the differential cross section in terms of all possible parity-parity combinations $i, j = V, A$ that occur through the virtual γ and Z states:

$$\begin{aligned}\frac{d\,\sigma}{d\cos\theta} &= \frac{N_C}{16}\left(\frac{\alpha}{q^2}\right)^2 v \sum_{i,j=V,A} g^{ij} \int d\varphi\, L^{\mu\nu}\, H^{ij}_{\mu\nu} \\ &= \frac{3}{8}\pi\left(\frac{\alpha}{q^2}\right)^2 v\left[g^{VV}\, L^{(\mu\nu)}\, H^{VV}_{(\mu\nu)} + g^{VA}\, L^{[\mu\nu]}\, H^{VA+AV}_{[\mu\nu]} + g^{AA}\, L^{(\mu\nu)}\, H^{AA}_{(\mu\nu)}\right], \qquad (2)\end{aligned}$$

where θ and φ are the the polar and azimuthal angles of the scattered quark. Here, L and H denote the lepton and hadron tensor, respectively, and as usual $N_C = 3$ takes into account that the produced quark pair comes in three colors. Because of cylindrical symmetry around the beam axis (as indicated in Fig. 2) the integration over φ is simplest. The mass parameter $v = \sqrt{1 - 4m^2/q^2}$ (m quark mass, q total energy-momentum transfer) enters Eq. (2) via the two-body phase-space factor $\mathrm{PS}_2 = v/8\pi$. Note that the product of lepton and hadron tensor with fixed parities $L^{\mu\nu}\, H^{ij}_{\mu\nu}$ is multiplied by the corresponding coefficients g^{ij}

$$g^{VV} = Q_q^2 - 2\, Q_q v_e v_q \,\mathrm{Re}\,\chi_Z + (v_e^2 + a_e^2)\, v_q^2\, |\chi_Z|^2, \qquad (3)$$

$$g^{VA} = -Q_q a_e a_q \,\mathrm{Re}\,\chi_Z + 2\, v_e a_e v_q a_q |\chi_Z|^2, \qquad (4)$$

$$g^{AA} = (v_e^2 + a_e^2)\, a_q^2\, |\chi_Z|^2, \qquad (5)$$

which contain the neutral-current couplings $v_f = 2\, T_z^f - 4\, Q_f \sin^2\theta_W$ and $a_f = 2\, T_z^f$, and Q_f denotes the fractional charge of the fermion. The Breit-Wigner form of the massive gauge boson is characterized by

$$\chi_Z(q^2) = \frac{g_F\, M_Z^2\, q^2}{q^2 - M_Z^2 + i M_Z\, \Gamma_Z} \quad \text{and} \quad g_F = \frac{G_F}{8\sqrt{2}\,\pi\alpha} \approx 4.299 \cdot 10^{-5}\,\mathrm{GeV}^{-2}. \qquad (6)$$

The normalization of the symmetric and antisymmetric components in the tensor decomposition

$$L^{\mu\nu} = L^{(\mu\nu)} + L^{[\mu\nu]} \qquad (7)$$

has to be in agreement with Eq. (2)

$$
\begin{aligned}
L^{(\mu\nu)} &= \frac{4}{q^2}\left(p_+^\mu p_-^\nu + p_+^\nu p_-^\mu - \tfrac{1}{2}q^2 g^{\mu\nu}\right), && (8)\\
L^{[\mu\nu]} &= \frac{4i}{q^2}\,\epsilon(\mu,\nu,p_-,p_+), && (9)
\end{aligned}
$$

where p_- and p_+ refer to the momenta of the electron and positron, respectively, and $q = p_- + p_+$ is the total momentum transfer as shown in Fig. 1b.

In Fig. 2, we show the kinematic configuration of the quark-antiquark final state in the center-of-momentum system (cms). The initial e^+e^- beam is aligned along the z-axis, so that the polar angle θ coincides with the scattering angle between the outgoing quark momentum $\boldsymbol{p_1}$ and the incoming electron momentum $\boldsymbol{p_-}$. As usual, the square of the cms energy is given by $q^2 = (p_1+p_2)^2 = (p_- + p_+)^2$. Recall that the remaining degree of freedom is captured by the rotation angle φ around the beam axis. In this coordinate frame, the particle momenta take a particularly simple contravariant form and the orientation of the quark momentum is given by $\hat{p} = \left(\sin\theta\,\cos\varphi,\, \sin\theta\,\sin\varphi,\, \cos\theta\right)$. Notice the meaning of $v = \sqrt{1-4m^2/q^2}$ as a dimensionless scale of the quark's velocity in the cms frame.

In the on-shell renormalization scheme the self-energy corrections vanish so that only the Feynman diagrams Figs. 1a and 1b are required to calculate the hadron tensor for the complete $\mathcal{O}(\alpha)$ process with two quarks $q(p_1)$ and $\bar{q}(p_2)$ in the final state. We choose the Feynman gauge for the internal gluon propagator. Then, the general substitutions made to arrive at the massive QCD vertices from the Born term expressions are simply

$$
\begin{aligned}
\gamma_\mu &\to (1+A)\,\gamma_\mu - B\,\frac{(p_1-p_2)_\mu}{2m}, && (10)\\
\gamma_\mu\gamma_5 &\to (1+C)\,\gamma_\mu\gamma_5 + D\,\frac{(p_1+p_2)_\mu}{2m}\,\gamma_5, && (11)
\end{aligned}
$$

with the induction of charge and magnetic moment form factors in the vector vertex and an axial-charge form factor in the axial vertex.

The explicit real parts of the chromomagnetic form factors are given by

$$
\begin{aligned}
\mathrm{Re}\,A &= \frac{\alpha_s}{4\pi}C_F\Bigg[\left(\frac{1+v^2}{v}\ln\left(\frac{1-v}{1+v}\right)+2\right)\Bigg(-\ln\Lambda + \ln(1-v^2) - 2\ln 2\Bigg)\\
&\qquad +F(v)\Bigg], && (12)
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Re} B &= \frac{\alpha_s}{4\pi} C_F \frac{1-v^2}{v} \ln\left(\frac{1-v}{1+v}\right), && (13)\\
\operatorname{Re} C &= \operatorname{Re} A - 2 \operatorname{Re} B, && (14)
\end{aligned}
$$

where Λ is a cut-off parameter to control the soft infrared divergences. At one-loop order, the cutoff Λ is related to the usual dimensional regulator $\varepsilon = 4 - N$ by the following correspondence rule [10]

$$
\ln \Lambda \longleftrightarrow \frac{2}{4-N} - \gamma_E + \ln\left(\frac{4\pi\mu^2}{q^2}\right), \tag{15}
$$

where γ_E is the Euler constant and μ the 't Hooft mass associated to the couplings in N dimensions. Furthermore, the shorthand $F(v)$ was introduced

$$
F(v) = \frac{1+v^2}{v}\left[\mathrm{Li}_2\left(\frac{2v}{v-1}\right) - \mathrm{Li}_2\left(\frac{2v}{v+1}\right) + \pi^2\right] + 3\,v \ln\left(\frac{1+v}{1-v}\right) - 4. \tag{16}
$$

As usual $C_F = 4/3$ is the Casimir operator defining the adjoint representation of the $\mathrm{SU}(3)_c$ color group. Note that the identity $C = A - 2\,B$ connects the gluonic vertex corrections of the vector and axial-vector currents for massive fermions.

The form factors Eqs. (12)–(14) have been derived in Ref. [4] with a completely anti-commuting Dirac γ_5 in dimensional regularization following the prescription of Chanowitz *et al.* [11]. The same final expressions [7,12] are also obtained by employing dimensional reduction. In this particular method, the Clifford algebra of the Dirac matrices is reduced to 4 dimensions while space-time is still kept in $N = 4 - \varepsilon$ dimensions to regularize the otherwise divergent loop integrals. At first glance, naive dimensional reduction and dimensional regularization seem to differ by a finite term. However, at one-loop order the inclusion of a global counterterm effectively restores the reduction in the spin degrees of freedom, so that both schemes are consistent with each other [12].

In dimensional reduction, a straightforward calculation now gives the $\mathcal{O}(\alpha)$ virtual corrections to the Born approximation. The key ingredients of Eq. (2) are

$$
\begin{aligned}
L^{(\mu\nu)}\, H^{VV}_{(\mu\nu)}(virtual) &= 4\,q^2 \Big[(1 + 2 \operatorname{Re} A)\left\{2 - v^2(1-\cos^2\theta)\right\} \\
&\qquad + 2 \operatorname{Re} B\, v^2 (1-\cos^2\theta) \Big], && (17)\\
L^{[\mu\nu]}\, H^{VA+AV}_{[\mu\nu]}(virtual) &= 16\, q^2 \,(1 + \operatorname{Re} A + \operatorname{Re} C)\, v\, \cos\theta, && (18)\\
L^{(\mu\nu)}\, H^{AA}_{(\mu\nu)}(virtual) &= 4\, q^2\, (1 + 2 \operatorname{Re} C)\, v^2 (1+\cos^2\theta). && (19)
\end{aligned}
$$

The soft IR divergences emerge in the QCD form factors A and C as the cut-off parameter $\Lambda \to 0$, whereas B gives only a finite correction. The imaginary part $i\,\mathrm{Im}\,D$ in $H^{AA}_{(\mu\nu)}$ can not contribute to the cross section and finally disappears because of leptonic current conservation. Note that the contractions of the symmetric lepton and hadron tensors Eqs. (17) and (19) contribute to the $\mathcal{C}$-even structure functions $\sigma_{L,U}$, whereas the contraction of the asymmetric tensors Eq. (18) yields σ_F which is $\mathcal{C}$-odd.

At this point, we stress that in the explicit $\mathcal{O}(\alpha)$ calculation of the $\mathcal{C}$-odd contribution σ_F, one faces immediately the well-known problem of how to extend the four-dimensional γ_5-matrix to N dimensions, or, in other words, how to treat the Lorentz indices of the totally skew-symmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ in N dimensions. Neither does dimensional regularization nor dimensional reduction avoid a direct confrontation with the γ_5-problem: in both cases space-time is N-dimensional. Moreover, the global reduction counterterm of Ref. [12] had been taken directly from the massless vector-current vertex. While its unrestricted usage for loops with even γ_5 has been demonstrated, there still remains to investigate its significance for odd γ_5 calculations.

On the other side, there exists a well-established technique based on the following substitution of the axial-vector current [13,14]

$$\gamma_\mu \gamma_5 \;\to\; Z_5 \,\frac{i}{3!}\, \epsilon_\mu{}^{\rho\sigma\tau}\, \gamma_\rho \gamma_\sigma \gamma_\tau \tag{20}$$

with

$$Z_5 = 1 - \frac{\alpha_s}{\pi}\, C_F + \mathcal{O}(\alpha). \tag{21}$$

The finite renormalization Z_5 is needed in addition to the conventional wavefunction renormalization Z_2 to restore anticommutativity [15] which is evidently violated after making the replacement Eq. (20). Although this replacement method has been developed in first place for massless multiloop calculations (see e.g. [16]), the generalization to the massive case is straightforward, since the renormalization of the axial current is related with the ultraviolet sector, whereas the collinear and soft divergences relate to the distinct infrared sector.

To complement our calculation, we examine the effects that this particular regularization scheme has on the gluonic correction of the axial-vector current. For this purpose,

we consider the contraction of the epsilon tensor with the chiral vacuum polarization tensor (with cuts in the two final fermion lines of the corresponding diagram). This gives the following definition of the C form factor:

$$
\begin{aligned}
C &= \frac{2\pi}{(q^2)^2 v^2}\,\alpha_s C_F\,\epsilon(\mu,\nu,p_1,p_2) \qquad (22)\\
&\times \int \frac{d^N k}{(2\pi)^N}\frac{\mathrm{Tr}\,(\not{p}_1+m)\,\gamma_\alpha\,(\not{p}_1+\not{k}+m)\,\gamma_\mu\gamma_5\,(-\not{p}_2+\not{k}+m)\,\gamma^\alpha\,(\not{p}_2-m)\,\gamma_\nu}{k^2\Big\{(p_1+k)^2-m^2\Big\}\Big\{(p_2-k)^2-m^2\Big\}},
\end{aligned}
$$

where k is the running momentum of the gluon loop.

Then, we can show that within the framework of the γ_5-replacement prescription Eq. (20) the previous C form factor Eq. (14) is recovered

$$
\begin{aligned}
\mathrm{Re}\,C &= \frac{\alpha_s}{4\pi}C_F\Bigg[2\bigg(\frac{1+v^2}{v}\ln\left(\frac{1+v}{1-v}\right)-2\bigg)\bigg(\ln\Lambda^{\frac{1}{2}}-\tfrac{1}{2}\ln(1-v^2)+\ln 2+1\bigg)\\
&\qquad -4v\ln\left(\frac{1+v}{1-v}\right)+F(v)+4\Bigg]. \qquad (23)
\end{aligned}
$$

In Appendix A, we outline the derivation of C in Eq. (23) with a special emphasis on the renormalization procedure and explicitly point out the differences to the dimensional reduction approach.

To obtain the result (23) we have used the $\mathcal{O}(\alpha)$ expression for the renormalization factor Z_5. The α_s-term in Eq. (21) is required to restore the usual chiral Ward identity corresponding to the conservation of the $\mathcal{O}(\alpha)$ axial current in the fermionic zero-mass limit.

With the γ_5-prescription of Eq. (20), we obtain the following expression for the $\mathcal{C}$-odd part in the differential cross section

$$
L^{[\mu\nu]}\,H^{VA+AV}_{[\mu\nu]}(virtual) = 4(4-3\varepsilon)\,q^2(1+\mathrm{Re}\,A+\mathrm{Re}\,C)v\cos\theta, \qquad (24)
$$

where the product of the two epsilon tensors has been replaced by the following natural extension of the four-dimensional identity to N dimensions

$$
\epsilon(\mu,\nu,\alpha,\beta)\epsilon(\mu,\nu,\rho,\sigma) \to -2\Big(\delta_{\alpha\rho}\delta_{\beta\sigma}-\delta_{\alpha\sigma}\delta_{\beta\rho}\Big). \qquad (25)
$$

Note that the ε-term in Eq. (24) gives additional finite contributions in the virtual part due to the infrared poles in A and C when compared with the result Eq. (18) obtained by dimensional reduction.

3 Real-Gluon Emission

The $\mathcal{O}(\alpha)$ tree-graph contributions of Fig. 1(c) are required to cancel the IR/M divergences in the virtual parts, and thus render the physical cross section finite in concordance with the Kinoshita-Lee-Nauenberg theorem [17]. In the following, we distinguish two types of IR/M singularities that occur at one-loop level in the real and virtual parts.

The *soft* divergences result from the massless character of gluon field and are regulated by introducing a small gluon mass m_g. The corresponding regulator $\Lambda = m_g^2/q^2$ is compatible with the cut-off definition in the form factors Eqs. (12) and (14), and defines the scale to discriminate the soft-gluon from the hard-gluon domain. In the case of real-gluon emission, $\Lambda \neq 0$ has the specific task to slightly deform the critical boundary region of the notorious three-body phase-space. On the other hand, the *collinear* divergences emerge when the gluon field couples with another massless field, and they typically manifest themselves as singularities in the mass parameter $\xi = 4m^2/q^2$.

Deriving the hadron tensors $h^{ij}_{\mu\nu}$ $(i,j = V,A)$ for the $\mathcal{O}(\alpha)$ tree-graph contributions, we only have to consider the lowest-order results in the gluon-mass expansion since the soft divergences, which will occur after performing the phase-space integration, are at most logarithmical, and thus vanish in the limit $\Lambda \to 0$ when multiplied by an additional Λ. Four-dimensional trace algebra gives the following explicit results for the parity-even contributions

$$\begin{aligned}
h^{VV}_{(\mu\nu)} &= 8\pi\,\alpha_s\,C_F\Bigg[\,g_{\mu\nu}\left\{\frac{\xi}{y^2}+\frac{4}{y}+\frac{\xi}{z^2}+\frac{4}{z}-2\frac{2-\xi}{yz}-2\frac{y}{z}-2\frac{z}{y}\right\} \\
&\qquad +4\,\frac{p_{1\mu}p_{1\nu}}{q^2}\left\{\frac{\xi}{y^2}-\frac{2}{yz}\right\}+4\,\frac{p_{2\mu}p_{2\nu}}{q^2}\left\{\frac{\xi}{z^2}-\frac{2}{yz}\right\} \\
&\qquad -4\,\frac{p_{1\mu}p_{2\nu}+p_{1\nu}p_{2\mu}}{q^2}\,\frac{\xi}{yz}\Bigg], \qquad (26) \\
h^{AA}_{(\mu\nu)} &= 8\pi\,\alpha_s\,C_F\Bigg[\,g_{\mu\nu}\Bigg\{4\frac{1-\xi}{y}+\xi\frac{1-\xi}{y^2}+4\frac{1-\xi}{z}+\xi\frac{1-\xi}{z^2} \\
&\qquad\qquad -2\frac{(1-\xi)(2-\xi)}{yz}-2\frac{y}{z}-2\frac{z}{y}\Bigg\} \\
&\qquad +4\,\frac{p_{1\mu}p_{1\nu}}{q^2}\left\{\frac{\xi}{y^2}-2\frac{1-\xi}{yz}\right\}+4\,\frac{p_{2\mu}p_{2\nu}}{q^2}\left\{\frac{\xi}{z^2}-2\frac{1-\xi}{yz}\right\} \\
&\qquad +4\,\frac{p_{1\mu}p_{2\nu}+p_{1\nu}p_{2\mu}}{q^2}\,\frac{\xi}{yz}\Bigg], \qquad (27)
\end{aligned}$$

where the phase-space variables

$$y = 1 - \frac{2\,p_1 \cdot q}{q^2} \qquad \text{and} \qquad z = 1 - \frac{2\,p_2 \cdot q}{q^2} \tag{28}$$

specify the energies carried by quark and antiquark in the cms frame. Energy-momentum conservation in combination with current conservation eliminated the gluon momentum p_3 and lead to drastic simplifications in these final expressions.

In the computation of the $\mathcal{C}$-odd contribution these favorable features are absent. Especially the γ_5-replacement prescription Eq. (20) yields quite lengthy expressions in the original trace

$$\begin{aligned} h^{VA}_{[\mu\nu]} &= h^{AV}_{[\mu\nu]}(real) \\ &= -\tfrac{1}{2}\, Tr(\not p_1 + m)\left[\gamma_\alpha \frac{\not p_1 + \not p_3 + m}{2\,p_1 \cdot p_3}\gamma_{[\mu} + \gamma_{[\mu} \frac{-\not p_2 - \not p_3 + m}{2\,p_2 \cdot p_3}\gamma_\alpha \right] \\ &\qquad (\not p_2 - m)\left[\gamma^\alpha \frac{-\not p_2 - \not p_3 + m}{2\,p_2 \cdot p_3}\gamma_{\nu]}\gamma_5 + \gamma_{\nu]}\gamma_5 \frac{\not p_1 + \not p_3 + m}{2\,p_1 \cdot p_3}\gamma^\alpha \right]. \end{aligned} \tag{29}$$

At this point, we shall therefore not reproduce the full expressions for the uncontracted hadron tensor $h^{VA}_{[\mu\nu]}$ but postpone further discussion of the $\mathcal{C}$-odd part until we will come to the final contracted results.

The $q\bar{q}g$ phase-space is considerably more complicated than the previously discussed two-particle complement. In general, five parameters are needed to completely characterize a specific configuration in three-particle phase-space. The phase-space parametrization is crucial for obtaining closed analytical solutions after integration out all variables except the $\cos\theta$-dependence and the cms energy. Our particular choice for these five parameters is $\sqrt{q^2}$, y, z, $\cos\theta$, and φ.

Energy-momentum conservation confines the momenta of the three outgoing particles to within a plane. Fig. 3 illustrates the angular orientation of the (p_1, p_2, p_3)-plane in the cms coordinate frame. The vector normal to the outgoing 3-jet plane is

$$\hat{n} = \Big(-\cos\theta\,\cos\varphi,\, -\cos\theta\,\sin\varphi,\, \sin\theta \Big), \tag{30}$$

and the direction of the scattered antiquark $\hat{p}_2$ is obtained by a rotation $\mathcal{R}$ of $\hat{p} = p_1/|p_1|$ around $\hat{n}$ with the angle χ. We find

$$\hat{p}_2 = \mathcal{R}_{\hat{n}}(\chi)\,\hat{p} = \hat{p}\,\cos\chi - \hat{n} \times \hat{p}\,\sin\chi. \tag{31}$$

Now it is straightforward to write all final momenta in terms of the chosen phase-space parameters (with the standard metric $g^{\mu\nu} = \mathrm{diag}(1; -1, -1, -1)$)

$$
\begin{aligned}
p_1 &= \tfrac{1}{2}\sqrt{q^2}\,\Big(1-y;\ \hat{p}\,\sqrt{(1-y)^2-\xi}\,\Big),\\
p_2 &= \tfrac{1}{2}\sqrt{q^2}\,\Big(1-z;\ \hat{p}_2\,\sqrt{(1-z)^2-\xi}\,\Big),\\
p_3 &= \tfrac{1}{2}\sqrt{q^2}\,(y+z)\,\Big(1;\ \hat{p}_3\Big),
\end{aligned}
\tag{32}
$$

where the gluon is scattered off in the direction

$$
\hat{p}_3 = -\hat{p}\,\frac{\sqrt{(1-y)^2-\xi}}{y+z} - \hat{p}_2\frac{\sqrt{(1-z)^2-\xi}}{y+z}. \tag{33}
$$

Note that $p_3^2 = 0$ automatically requires $\hat{p}_3$ to have unit length which yields in combination with Eq. (31)

$$
\cos\chi = \hat{p}\cdot\hat{p}_2 = \frac{y+z+yz+\xi-1}{\sqrt{(1-y)^2-\xi}\sqrt{(1-z)^2-\xi}}, \tag{34}
$$

which makes evident that χ gives no additional degree of freedom but is fully determined by energy-momentum conservation. It is now easy to obtain the following relations to classify the angular dependence that the various hadronic components of Eqs. (26) and (27) give upon contraction with the lepton tensor

$$
\begin{aligned}
L^{\mu\nu}\,g_{\mu\nu} &= -4, &(35)\\
L^{\mu\nu}\,\frac{p_{1\mu}p_{1\nu}}{q^2} &= \tfrac{1}{2}\sin^2\theta\,\Big\{\,(1-y)^2-\xi\,\Big\}, &(36)\\
L^{\mu\nu}\,\frac{p_{2\mu}p_{2\nu}}{q^2} &= \tfrac{1}{2}\Big(1-\cos^2\chi\,\cos^2\theta\Big)\,\Big\{\,(1-z)^2-\xi\,\Big\}, &(37)\\
L^{\mu\nu}\,\frac{p_{1\mu}p_{2\nu}+p_{1\nu}p_{2\mu}}{q^2} &= \tfrac{1}{2}\sin^2\theta\,\Big\{\,y+z+yz+\xi-1\,\Big\}. &(38)
\end{aligned}
$$

Hence, we readily find the following contracted results after substituting for $\cos\chi$ according to Eq. (34)

$$
\begin{aligned}
&L^{\mu\nu}\,h^{VV}_{\mu\nu} = 8\pi\,\alpha_s\,C_F\,\Bigg[\\
&-8(1+\xi)\left(\frac{1}{y}+\frac{1}{z}\right) - 2\xi(1+\xi)\left(\frac{1}{y^2}+\frac{1}{z^2}\right) + 4\left(\frac{y}{z}+\frac{z}{y}\right) + 4(1+\xi)(2-\xi)\frac{1}{yz}\\
&+\frac{2\cos\theta}{(1-y)^2-\xi}\Bigg\{-5\xi - 2(1-\xi)(2-3\xi)\frac{1}{y} - \xi(1-\xi)^2\frac{1}{y^2} - 4(1-\xi)(3-2\xi)\frac{1}{z}\\
&\qquad -\xi(1-\xi)^2\frac{1}{z^2} - 2\xi(1-\xi)\frac{y}{z^2} - \xi\frac{y^2}{z^2} + 4z + 2\frac{z}{y} + 2(7-5\xi)\frac{y}{z}
\end{aligned}
$$

$$+8\frac{y^2}{z}+2(2-\xi)(1-\xi)^2\frac{1}{yz}+2\frac{y^3}{z}+4y+2yz\Big\}\;\Big], \tag{39}$$

and

$$L^{\mu\nu}\,h^{AA}_{\mu\nu}=8\pi\,\alpha_s\,C_F\Big[$$
$$8\xi-8(1-\xi))\left(\frac{1}{y}+\frac{1}{z}\right)-2\xi(1-\xi)\left(\frac{1}{y^2}+\frac{1}{z^2}\right)+4(1+\xi)\left(\frac{y}{z}+\frac{z}{y}\right)$$
$$+4(1-\xi)(2-\xi)\frac{1}{yz}+\frac{2\cos\theta}{(1-y)^2-\xi}\Big\{-\xi-2(1-\xi)(2-3\xi)\frac{1}{y}-\xi(1-\xi)^2\frac{1}{y^2}$$
$$-4(1-\xi)(3-2\xi)\frac{1}{z}-\xi(1-\xi)^2\frac{1}{z^2}+2\xi(1-\xi)\frac{y}{z^2}+2(1-\xi)\frac{y^3}{z}-\xi\frac{y^2}{z^2}+4(1-\xi)z$$
$$+2(1-\xi)\frac{z}{y}+2(7-6\xi)\frac{y}{z}-4(2-\xi)\frac{y^2}{z}+2(2-\xi)(1-\xi)^2\frac{1}{yz}+4y-4\xi y^2$$
$$+2(1-\xi)yz\Big\}\;\Big]. \tag{40}$$

Out of the five variables that parametrize the three-particle phase-space we only need to remove the y- and z-dependence by integration to yield the differential cross section, depending solely on $E_{cms}=\sqrt{q^2}$ and $\cos\theta$. Note that in Eq. (2) the φ-integration has already been included in the two-particle phase-space factor PS_2 apart from the flux factor. Thus, the appropriate (y,z)-integration can be put into the form

$$H^{ij}_{\mu\nu}(real)=\int\frac{d^4\mathrm{PS}_3}{\mathrm{PS}_2}\,h^{ij}_{\mu\nu}=\frac{q^2}{16\pi^2\,v}\int\limits_{y_-}^{y_+}dy\int\limits_{z_-(y)}^{z_+(y)}dz\,h^{ij}_{\mu\nu}, \tag{41}$$

which gives the final real-gluon contributions that have to be included in Eq. (2) for the cancellation of all IR divergences with the corresponding virtual parts.

The complicated upper and lower bounds of the nested integral in Eq. (41) are given in Appendix B where we discuss in more detail the $q\bar{q}g$ phase-space. In our scheme, the phase-space integrals are grouped into distinct classes $\{\mathcal{I}_i\},\{\mathcal{S}_i\}$, and $\{\mathcal{J}_i\}$. Each class refers to a different functional dependence on the quark's velocity in the three-body system. The individual elements of $\{\mathcal{I}_i\},\{\mathcal{S}_i\}$, and $\{\mathcal{J}_i\}$ are identified in Appendix B which also treats some specific properties of the integrals and their interconnection.

Thus, the final integrated results for the $\mathcal{O}(\alpha)$ tree graphs with VV and AA parity-parity combinations are

$$L^{\mu\nu}\,H^{VV}_{\mu\nu}(real)=\frac{q^2}{2v}\,\frac{\alpha_s}{\pi}\,C_F\Big[-16(1+\xi)\mathcal{I}_2-4\xi(1+\xi)\mathcal{I}_3+8\mathcal{I}_4+4(1+\xi)(2-\xi)\mathcal{I}_5$$

$$+2\cos^2\theta\Big\{ -8\mathcal{I}_2 - 5\xi\mathcal{J}_1 - 2(1-\xi)(2-3\xi)\mathcal{J}_2 - \xi(1-\xi)^2\mathcal{J}_3 + 4\mathcal{J}_4 + 2\mathcal{J}_6 + 4\mathcal{J}_7$$

$$-4(1-\xi)(1-2\xi)\mathcal{J}_8 - \xi(1-\xi)^2\mathcal{J}_9 + 2(2-\xi)(1-\xi)^2\mathcal{J}_{10} + 2\mathcal{J}_{11} - 2(1+5\xi)\mathcal{J}_{12}$$

$$+2\xi(1-\xi)\mathcal{J}_{13} - \xi\mathcal{J}_{14} + 2\mathcal{J}_{15}\Big\}\ \Big], \tag{42}$$

$$L^{\mu\nu}\, H^{AA}_{\mu\nu}(real) = \frac{q^2}{2v}\,\frac{\alpha_s}{\pi}\,C_F\Big[8\xi\mathcal{I}_1 - 16(1-\xi)\mathcal{I}_2 - 4\xi(1-\xi)\mathcal{I}_3 + 8(1+\xi)\mathcal{I}_4$$

$$+4(1-\xi)(2-\xi)\mathcal{I}_5 + 2\cos^2\theta\Big\{ -4(2-\xi)\mathcal{I}_2 - \xi\mathcal{J}_1 - 2(1-\xi)(2-3\xi)\mathcal{J}_2 - \xi(1-\xi)^2\mathcal{J}_3$$

$$+4\mathcal{J}_4 - 4\xi\mathcal{J}_5 + 2(1-\xi)\mathcal{J}_6 + 4(1-\xi)\mathcal{J}_7 - 4(1-\xi)^2\mathcal{J}_8 - \xi(1-\xi)^2\mathcal{J}_9 + 2(2-\xi)(1-\xi)^2\mathcal{J}_{10}$$

$$+2(1-\xi)\mathcal{J}_{11} - 2(1+2\xi)\mathcal{J}_{12} + 2\xi(1-\xi)\mathcal{J}_{13} - \xi\mathcal{J}_{14} + 2(1-\xi)\mathcal{J}_{15}\Big\}\ \Big]. \tag{43}$$

Note that $\{\mathcal{I}_i\}, \{\mathcal{S}_i\}$, and $\{\mathcal{J}_i\}$ are the smallest units of the three-particle phase-space one arrives at after parametrization in the energy variables (y, z) and subsequent partial fractioning. These single components are process-independent. With the additional integral class $\mathcal{T}_i$ (see Ref. [9]) they constitute a complete set of solutions for any one-loop real-emission process from massive fermions. Differential or total observables such as production rates or various polarization components can all be expressed succinctly in terms of these units once they are derived.

To contract the asymmetric part of the lepton tensor $L^{[\mu\nu]}$ with the γ_5-odd hadronic complement Eq. (29), we use the following identities

$$L^{\mu\nu}\,\frac{\epsilon(\mu,\nu,p_1,q)}{q^2} = i\,2v\,\cos\theta, \tag{44}$$

$$L^{\mu\nu}\,\frac{\epsilon(\mu,\nu,p_2,q)}{q^2} = -i\,2v\,\cos\theta, \tag{45}$$

$$L^{\mu\nu}\,\frac{\epsilon(\mu,\nu,p_1,p_2)}{q^2} = -i\,\frac{2\cos\theta}{\sqrt{(1-y)^2-\xi}}\Big\{ (2-\tfrac{1}{2}\xi)y + (1-\tfrac{1}{2}\xi)z - y^2 - yz + \xi - 1\Big\}, \tag{46}$$

$$L^{\mu\nu}\,\frac{\epsilon(\mu,p_1,p_2,q)\,p_{2\nu} - \epsilon(\nu,p_1,p_2,q)\,p_{2\mu}}{(q^2)^2} = i\,v\,\cos\theta\,y, \tag{47}$$

$$L^{\mu\nu}\,\frac{\epsilon(\mu,p_1,p_2,q)\,q_\nu - \epsilon(\nu,p_1,p_2,q)\,q_\mu}{(q^2)^2} = i\,v\,\cos\theta\,(y+z). \tag{48}$$

In these expressions, the contraction of two epsilon tensors with one common index has

been replaced by an appropriately antisymmetrized product of metric tensors which is well-defined in either four or N dimensions:

$$\epsilon_{\mu\alpha\beta\gamma}\,\epsilon^{\mu\rho\sigma\tau} = -3!\,g_\alpha^{\;[\rho}\,g_\beta^{\;\sigma}\,g_\gamma^{\;\tau]}. \tag{49}$$

Then, the scalar product of two Lorentz vectors is invariant in different space-time dimensions so that Eqs. (44)–(48) give unique results.

It is now straightforward to obtain the following contracted results for the general case of N-dimensional Clifford algebra

$$\begin{aligned} & L^{\mu\nu}\,h^{VA}_{\mu\nu} = L^{\mu\nu}\,h^{AV}_{\mu\nu} \\ &= 32\pi\,\alpha_s\,C_F\,\cos\theta\,\left(1-\tfrac{3}{4}\varepsilon\right)\Bigg[\varepsilon\xi-(4-5\xi)\frac{1}{y}-\xi(1-\xi)\left(\frac{1}{y^2}+\frac{1}{z^2}\right) \\ &\quad -2(4-3\xi)\frac{1}{z}+\xi\frac{y}{z^2}+2\varepsilon y+(2+\varepsilon)z+\left\{2-\varepsilon\left(1-\tfrac{1}{2}\xi\right)\right\}\frac{z}{y} \\ &\quad +\left\{6-\varepsilon\left(1-\tfrac{1}{2}\xi\right)\right\}\frac{y}{z}-(2-\varepsilon)\frac{y^2}{z}+(1-\xi)(2-\xi)\frac{1}{yz}+\mathcal{O}(\varepsilon)\Bigg]. \end{aligned} \tag{50}$$

Putting ε in Eq. (50) to zero and then performing the (y,z)-integration gives the following result for the $\mathcal{C}$-odd tree-graph contribution within dimensional reduction

$$\begin{aligned} & L^{\mu\nu}\,H^{VA}_{\mu\nu}(real) = L^{\mu\nu}\,H^{AV}_{\mu\nu}(real) \\ &= 2\,\frac{q^2}{v}\,\frac{\alpha_s}{\pi}\,C_F\,\cos\theta\Bigg[-(4-5\xi)\mathcal{S}_2-\xi(1-\xi)(\mathcal{S}_3+\mathcal{S}_5)-2(4-3\xi)\mathcal{S}_4 \\ &\quad +\xi\mathcal{S}_6+2\mathcal{S}_8+2\mathcal{S}_9+6\mathcal{S}_{10}-2\mathcal{S}_{11}+2(1-\xi)(2-\xi)\mathcal{S}_{12}\Bigg]. \end{aligned} \tag{51}$$

On the other hand, adopting the γ_5-prescription of Eq. (20) we obtain

$$\begin{aligned} & L^{\mu\nu}\,H^{VA}_{\mu\nu}(real) = L^{\mu\nu}\,H^{AV}_{\mu\nu}(real) \\ &= 2\,\frac{q^2}{v}\,\frac{\alpha_s}{\pi}\,C_F\,\cos\theta\,\left(1-\tfrac{3}{4}\varepsilon\right)\Bigg[\varepsilon\xi\mathcal{S}_1-(4-5\xi)\mathcal{S}_2-\xi(1-\xi)(\mathcal{S}_3+\mathcal{S}_5)-2(4-3\xi)\mathcal{S}_4 \\ &\quad +\xi\mathcal{S}_6+(2+\varepsilon)\mathcal{S}_8+\left\{2-\varepsilon\left(1-\tfrac{1}{2}\xi\right)\right\}\mathcal{S}_9+\left\{6-\varepsilon\left(1-\tfrac{1}{2}\xi\right)\right\}\mathcal{S}_{10} \\ &\quad -(2-\varepsilon)\mathcal{S}_{11}+2(1-\xi)(2-\xi)\mathcal{S}_{12}+2\varepsilon\mathcal{S}_{13}+\mathcal{O}(\varepsilon)\Bigg] \end{aligned} \tag{52}$$

$$= \; 2\,\frac{q^2}{v}\,\frac{\alpha_s}{\pi}\,C_F\,\cos\theta\Bigg[-(4-5\xi)\mathcal{S}_2-(1-\tfrac{3}{4}\varepsilon)\xi(1-\xi)(\mathcal{S}_3+\mathcal{S}_5)-2(4-3\xi)\mathcal{S}_4$$

$$+\xi\mathcal{S}_6+2\mathcal{S}_8+2\mathcal{S}_9+6\mathcal{S}_{10}-2\mathcal{S}_{11}+2(1-\tfrac{3}{4}\varepsilon)(1-\xi)(2-\xi)\mathcal{S}_{12}+\mathcal{O}(\varepsilon)\Bigg]. \qquad (53)$$

Eq. (52) displays the full result neglecting $\mathcal{O}(\varepsilon)$ bracket terms, which do not contribute in the limit $\varepsilon \to 0$. Since the logarithmic divergences within the massive gluon scheme correspond at $\mathcal{O}(\alpha)$ to the $1/\varepsilon$ poles of a dimensionally regularized theory, we can not naively discard all $\mathcal{O}(\varepsilon)$-terms. In Eq. (53) only the non-vanishing $\mathcal{O}(\varepsilon)$ bracket terms are retained. However, these additional $\mathcal{O}(\varepsilon)$ terms represent the finite difference in the γ_5-odd real-gluon correction between dimensional reduction and dimensional regularization with the γ_5-replacement scheme. Finally, when the corresponding virtual-gluon correction Eq. (24) is added, these contributions disappear, and one obtains the same total result as in dimensional reduction Eq. (51).

4 Complete $\mathcal{O}(\alpha)$ Differential Cross Sections

With the explicit expressions for the virtual and real contributions of the previous sections, we are now in the position to give the full analytical results for the $\mathcal{O}(\alpha)$ differential cross section of the annihilation process $e^+e^- \to \gamma, Z \to q\bar{q}$. Full quark-mass dependence was kept and no other approximations were made so that these analytical formulae are valid in a closed form over the entire energy range of perturbative QCD. All angular dependence resulted naturally from basic kinematics of the two- and three-body final state.

A severe test on the total results is provided by the cancellation of the soft and collinear divergences separately for each angular distribution within the corresponding real and virtual parity-parity combinations. Using the relative phase-space factor Eq. (41) in the sum of virtual and real gluon parts, we obtain the following total $\mathcal{O}(\alpha)$ results (including the Born level)

$$L^{\mu\nu}\,H^{VV}_{\mu\nu}(total) = 4q^2\Bigg[(1+2\,\mathrm{Re}\,\tilde{A})\Big\{2-v^2(1-\cos^2\theta)\Big\}+2\,\mathrm{Re}\,B\,v^2(1-\cos^2\theta)$$

$$+\frac{\alpha_s}{4\pi}\,C_F\,\frac{1}{v}\Bigg\{-8(1+\xi)\mathcal{I}_2-2\xi(1+\xi)\tilde{\mathcal{I}}_3+4\mathcal{I}_4+2(1+\xi)(2-\xi)\tilde{\mathcal{I}}_5$$

$$+\cos^2\theta\Big\{-8\mathcal{I}_2-5\xi\mathcal{J}_1-2(1-\xi)(2-3\xi)\mathcal{J}_2-\xi(1-\xi)^2\tilde{\mathcal{J}}_3+4\mathcal{J}_4+2\mathcal{J}_6+4\mathcal{J}_7$$

$$-4(1-\xi)(1-2\xi)\mathcal{J}_8-\xi(1-\xi)^2\tilde{\mathcal{J}}_9+2(2-\xi)(1-\xi)^2\tilde{\mathcal{J}}_{10}+2\mathcal{J}_{11}-2(1+5\xi)\mathcal{J}_{12}$$

$$+2\xi(1-\xi)\mathcal{J}_{13}-\xi\mathcal{J}_{14}+2\mathcal{J}_{15}\Big\}\Bigg\}\Bigg], \tag{54}$$

$$L^{\mu\nu}\,H^{AA}_{\mu\nu}(total)=4q^2\Bigg[(1+2\,\mathrm{Re}\,\tilde{C})\,v^2(1+\cos^2\theta)$$

$$+\frac{\alpha_s}{4\pi}\,C_F\,\frac{1}{v}\Bigg\{4\xi\mathcal{I}_1-8(1-\xi)\mathcal{I}_2-2\xi(1-\xi)\tilde{\mathcal{I}}_3+4(1+\xi)\mathcal{I}_4+2(1-\xi)(2-\xi)\tilde{\mathcal{I}}_5$$

$$+\cos^2\theta\Big\{-4(2-\xi)\mathcal{I}_2-\xi\mathcal{J}_1-2(1-\xi)(2-3\xi)\mathcal{J}_2-\xi(1-\xi)^2\tilde{\mathcal{J}}_3+4\mathcal{J}_4-4\xi\mathcal{J}_5$$

$$+2(1-\xi)\mathcal{J}_6+4(1-\xi)\mathcal{J}_7-4(1-\xi)^2\mathcal{J}_8-\xi(1-\xi)^2\tilde{\mathcal{J}}_9+2(2-\xi)(1-\xi)^2\tilde{\mathcal{J}}_{10}$$

$$+2(1-\xi)\mathcal{J}_{11}-2(1+2\xi)\mathcal{J}_{12}+2\xi(1-\xi)\mathcal{J}_{13}-\xi\mathcal{J}_{14}+2(1-\xi)\mathcal{J}_{15}\Big\}\Bigg\}\Bigg], \tag{55}$$

where the wiggle on top of form factors or integrals denotes quantities free from soft divergences as $\Lambda\to 0$, which essentially means that only linear terms in $\ln\Lambda$ have been discarded. We find that in the massless limit $\xi\to 0$ all collinear singularities cancel, as expected. It is now easy to obtain the full $\mathcal{O}(\alpha)$ expressions for the structure functions in Eq. (1) by using the convolutions

$$\sigma_{U,L}=\frac{3}{8}\pi\left(\frac{\alpha}{q^2}\right)^2 v\sum_{ij=VV,AA} g^{ij}\,\Pi_{U,L}\,L^{(\mu\nu)}\,H^{ij}_{(\mu\nu)}, \tag{56}$$

where the unpolarized and longitudinal projectors are explicitly given by

$$\Pi_U \;=\; \int_{-1}^{+1} d\cos\theta\,\Big\{5\cos^2\theta-1\Big\}, \tag{57}$$

$$\Pi_L \;=\; \int_{-1}^{+1} d\cos\theta\,\Big\{2-5\cos^2\theta\Big\}. \tag{58}$$

Similarly, one finds that the remaining total VA combination is IR finite. Adding Eqs. (24) and (53) gives for the $\mathcal{C}$-odd component of the differential rate

$$\sigma_F=8\pi\,\frac{\alpha^2}{q^2}\,v\,g^{VA}\Bigg[(1+\mathrm{Re}\,\tilde{A}+\mathrm{Re}\,\tilde{C})\,v$$

$$+\frac{\alpha_s}{4\pi}\,C_F\,\frac{1}{v}\Bigg\{-(4-5\xi)\mathcal{S}_2-\xi(1-\xi)(\tilde{\mathcal{S}}_3+\tilde{\mathcal{S}}_5)-2(4-3\xi)\mathcal{S}_4+\xi\mathcal{S}_6$$

$$+2(\mathcal{S}_8+\mathcal{S}_9+3\mathcal{S}_{10}-\mathcal{S}_{11})+2(1-\xi)(2-\xi)\tilde{\mathcal{S}}_{12}\Big\}\Bigg], \tag{59}$$

where we substituted A and C according to Eqs. (12) and (23) without soft divergences. Note that for the N-dimensional γ_5-replacement scheme the terms proportional to ε in the real part have exactly canceled with the extra contributions in the virtual part. Thus, dimensional reduction and dimensional regularization give the same result for the forward-backward asymmetry in the differential cross section. The independent result for dimensional reduction is easily obtained by adding Eqs. (18) and (51).

In the massless calculation, there are no QCD one-loop contributions to the forward-backward asymmetry [4,22]. Apart from the cancellation of the spurious IR divergences in Eq. (59), we recover this specific result in the limit $\xi \to 0$. On the other hand, we have checked explictly that integrating the VV and AA parts of the differential rate over $\cos\theta$ reproduces exactly the analytical results for the total cross section as found in Ref. [3].

The numerical estimates of the $\mathcal{O}(\alpha)$ differential cross section for bottom quark production with a fixed-point mass $m_b(m_b) = 4.3$ GeV are shown in Fig. 4. The running of the strong coupling is implemented by taking $\alpha_s^{(5)}(M_Z) = 0.123$ in the modified minimal subtraction scheme ($\overline{\rm MS}$) for five active flavors. For the bottom quark at energies above the next flavor threshold and in general for the top quark, we apply the appropriate matching conditions for six active flavors using the corresponding one-loop QCD renormalization group equations [18].

Fig. 4a gives a surface plot of the $\mathcal{O}(\alpha)$ differential rate as a function of the cms energy $E_{cms} = \sqrt{q^2}$ and the cosine of the scattering angle $\cos\theta$. On the Z-peak, the asymmetry is clearly pronounced and yields for $E_{cms} = M_Z = 91.178$ GeV values ranging from 5800 pb ($\theta = \pi$) to 8300 pb ($\theta = 0$). The minimum is located at $\cos\theta \approx 0.16$ with 3494 pb. For higher energies the differential rate falls rapidly off to give at 100 GeV cross sections of the order of only 100 pb. Nevertheless, the massive $\mathcal{O}(\alpha)$ corrections become increasingly important off the Z-peak. In Fig. 4b, the $\mathcal{O}(\alpha)$ results are compared with the Born approximation. Note that the dominant $\mathcal{O}(\alpha)$ contributions are below the Z-peak in the domain $0 < \cos\theta < 0.4$ and above the Z-peak in the domain $-0.5 < \cos\theta < 0$. Thus, the area of maximum correction (4.3–4.5 %) consists of a strip of approximately $\pi/6$ width that shifts from the upper to the lower hemisphere when passing the Z-threshold

(see also Fig. 2). At 100 GeV the corrections amount for $\cos\theta = 0.3$ to full $4.5\,\%$.

Fig. 5a depicts a similar three-dimensional plot with the differential cross section for $e^+e^- \to \gamma, Z \to t\bar{t}$ at one-loop QCD level. The value for the top mass is $m_t(m_t) = 174\,\mathrm{GeV}$. Note that we have chosen an energy range sufficiently higher than the $t\bar{t}$ threshold to avoid full interference with the non-perturbative sector. The dominant contribution to the differential rate is given by top quarks scattered in the forward direction. Along the forward direction, we find at $\sim 425\,\mathrm{GeV}$ a peak value of ~ 0.85 pb for the cross section which includes an $\mathcal{O}(\alpha)$ correction of nearly $20\,\%$. At 350 GeV the strong corrections have an impact of $52\,\%$. Thus, it is conceivable that for top quark production non-perturbative effects still prevail in energy domains considerably above the $t\bar{t}$ threshold.

In Fig. 5b, we give the distribution of the energy and scattering regions most significant to the strong corrections. Note that the $\mathcal{O}(\alpha)$ terms contribute dominantly along the forward-scattering axis where the differential rate takes maximum values. On the other hand, bottom production yielded the most important $\mathcal{O}(\alpha)$ corrections perpendicular to the beam axis which corresponds to minimal values on the saddle surface in Fig. 4a.

5 Conclusions

In the present work, we have derived from first principles the full analytical $\mathcal{O}(\alpha)$ results for the differential cross sections in heavy-quark production. No mass approximations or further restrictive assumptions were made. All angular distributions arose naturally from the underlying phase-space kinematics.

The systematic treatment of the tree-graph contributions relies on the exact integral solutions of the massive three-body phase-space. These integrals are divided into several classes according to their functional dependence on the final quark velocity. A complete collection of these phase-space integrals allows to describe any one-loop bremsstrahlung process including mass effects.

In particular, we put emphasis on the consistent treatment of the axial-vector current to handle the spurious chiral anomalies that are usually present in dimensionally

regularized calculations with an odd number of γ_5's. We explicitly showed that dimensional reduction and dimensional regularization with the γ_5-replacement prescription yield identical results for the forward-backward asymmetry.

In our numerical estimates, we found that at one-loop level the final state QCD corrections already modify the Born approximation by approximately 4.5 % (100 GeV) for bottom production and more than 25 % (375 GeV) for top production. We presented details on the differential distribution, and made the essential observation that for the bottom quark the $\mathcal{O}(\alpha)$ corrections are dominant in the scattering region perpendicular to the beam axis whereas for top quarks $\mathcal{O}(\alpha)$ contributions become very important along the forward-scattering axis.

We conclude that full analytical calculations for heavy-fermion pair production at one-loop order are feasible and provide an attractive alternative to existing Monte-Carlo generators.

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Appendix A: Renormalization of the Axial-Vector Current

In this appendix, we give a brief account of the subtleties associated with the renormalization of the axial-vector current at QCD one-loop level.

As we explained in Section 2, dimensional regularization with a conventional anti-commuting γ_5 or a replaced γ_5 produces identical results for the gluonic vertex correction to the axial-vector current. Explicitly, we gave meaning to the γ_5-matrix in a dimension other than four by the replacement rule

$$\gamma_\mu\gamma_5 \;\rightarrow\; Z_5\,\frac{i}{3!}\,\epsilon_\mu{}^{\rho\sigma\tau}\,\gamma_\rho\gamma_\sigma\gamma_\tau,$$

where the finite renormalization constant Z_5 is necessary to reinstate the validity of the axial-vector Ward identities in the final results [15,16].

Using Eq. (49) one can recast Eq. (22) into the following expression for the unrenormalized form factor

$$\begin{aligned} C_{\text{bare}} &= -8\frac{\pi i}{3N-8}\,\alpha_s C_F\,\frac{g^{\nu\,[\rho}\,p_1^\sigma\,p_2^{\tau\,]}}{v^2(q^2)^2} \\ &\times \int\frac{d^Nk}{(2\pi)^N}\frac{\text{Tr}\,(\not{p}_1+m)\,\gamma_\alpha\,(\not{p}_1+\not{k}+m)\,\gamma_\rho\gamma_\sigma\gamma_\tau\,(-\not{p}_2+\not{k}+m)\,\gamma^\alpha\,(\not{p}_2-m)\,\gamma_\nu}{k^2\left\{(p_1+k)^2-m^2\right\}\left\{(p_2-k)^2-m^2\right\}}, \end{aligned} \tag{A1}$$

where all indices are N-dimensional. Note that the normalization has to be in agreement with the overall N-dependent factor in the Born contribution Eq. (24).

In the solution of this one-loop integral we use the following structure for the two- and three-point functions ($p_1^2 = p_2^2 = m^2$)

$$I_3(p_1,p_2) = \int\frac{d^Nk}{(2\pi)^N}\frac{1}{k^2\left\{(p_1+k)^2-m^2\right\}\left\{(p_2-k)^2-m^2\right\}} = \frac{P(v)}{m^2}, \tag{A2}$$

$$\begin{aligned} I_3^\mu(p_1,p_2) &= \int\frac{d^Nk}{(2\pi)^N}\frac{k^\mu}{k^2\left\{(p_1+k)^2-m^2\right\}\left\{(p_2-k)^2-m^2\right\}} \\ &= Q(v)\frac{(p_1-p_2)^\mu}{m^2}, \end{aligned} \tag{A3}$$

$$I_3^{\mu\nu}(p_1,p_2) = \int\frac{d^Nk}{(2\pi)^N}\frac{k^\mu k^\nu}{k^2\left\{(p_1+k)^2-m^2\right\}\left\{(p_2-k)^2-m^2\right\}} \tag{A4}$$

$$= R(v)\, g^{\mu\nu} + S(v)\frac{(p_1-p_2)^\mu (p_1-p_2)^\nu}{m^2} + T(v)\frac{(p_1+p_2)^\mu (p_1+p_2)^\nu}{q^2},$$

$$I_2(p_1,p_2) = \int \frac{d^N k}{(2\pi)^N} \frac{1}{\left\{(p_1+k)^2-m^2\right\}\left\{(p_2-k)^2-m^2\right\}} = g_{\mu\nu} I_3^{\mu\nu}(p_1,p_2). \quad \text{(A5)}$$

Notice that the last identity only holds if one takes $g^{\mu\nu}\, g_{\mu\nu} = N = 4-\varepsilon$. All coefficient functions $P,\ldots,T$ are dimensionless and depend only on the mass parameter v.

An explicit calculation yields for Eq. (A1)

$$\begin{aligned} C_{\text{bare}} = {} & 16\pi i\, \alpha_s C_F \left(1+\tfrac{3}{4}\varepsilon\right)\Bigg[\left(1-\tfrac{3}{4}\varepsilon\right)\frac{1+v^2}{1-v^2}\Big\{P(v)+2Q(v)\Big\} - \left(1+\tfrac{1}{4}\varepsilon\right)R(v) \\ & + \left(1-\tfrac{1}{4}\varepsilon\right)\frac{2v^2}{1-v^2}S(v) - \tfrac{1}{8}(4-\varepsilon)T(v) + \mathcal{O}(\varepsilon)\Bigg]. \quad \text{(A6)} \end{aligned}$$

Here, the ε-dependence stems exclusively from the N-dimensional trace algebra so that the result for dimensional reduction is obtained from the above formula by putting $\varepsilon = 0$. We verified this result also in an independent calculation.

Adopting the conventions of Refs. [13,20], we can further decompose the elements of Eqs. (A2)–(A5) into the so-called Passarino-Veltman functions

$$P(v) = \frac{i\mu^{-\varepsilon}}{(4\pi)^2} m^2\, C_0\left(m^2, q^2, m^2; 0, m^2, m^2\right), \quad \text{(A7)}$$

$$Q(v) = \frac{i\mu^{-\varepsilon}}{(4\pi)^2}\frac{1-v^2}{4v^2}\Bigg[B_0\left(q^2; m^2, m^2\right) - B_0\left(m^2; 0, m^2\right) \Bigg], \quad \text{(A8)}$$

$$R(v) = \frac{i\mu^{-\varepsilon}}{4(4\pi)^2}\frac{1-v^2}{4v^2}\Bigg[1 + B_0\left(q^2; m^2, m^2\right) \Bigg], \quad \text{(A9)}$$

$$S(v) = \frac{i\mu^{-\varepsilon}}{(4\pi)^2}\frac{1-v^2}{16v^2}\Bigg[B_0\left(m^2; 0, m^2\right) - B_0\left(q^2; m^2, m^2\right) \Bigg], \quad \text{(A10)}$$

$$T(v) = \frac{i\mu^{-\varepsilon}}{4(4\pi)^2}\Bigg[B_0\left(m^2; 0, m^2\right) - B_0\left(q^2; m^2, m^2\right) - 2 \Bigg]. \quad \text{(A11)}$$

For conciseness, we do not consider the imaginary contributions to the form factor but concentrate on the real parts of the following general scalar integral representations

$$B_0\left(p_1^2; m_0^2, m_1^2\right) = (i\,\pi^2)^{-1}\int d^N k\, \frac{1}{\left\{k^2-m_0^2\right\}\left\{(k+p_1)^2-m_1^2\right\}}, \quad \text{(A12)}$$

$$C_0\left(p_1^2, (p_1-p_2)^2, p_2^2; m_0^2, m_1^2, m_2^2\right) =$$

$$(i\,\pi^2)^{-1}\int d^N k\, \frac{1}{\left\{k^2-m_0^2\right\}\left\{(k+p_1)^2-m_1^2\right\}\left\{(k+p_2)^2-m_2^2\right\}}, \quad \text{(A13)}$$

In the required Passarino-Veltman functions we neglect terms proportional to ε and drop the imaginary parts. These scalar two- and three-point functions are divergent as $\varepsilon \to 0$ and read

$$
\begin{aligned}
B_0\left(m^2;0,m^2\right) &= \frac{2}{\varepsilon} - \gamma_{\scriptscriptstyle E} + \ln\left(\frac{4\pi\mu^2}{m^2}\right) + 2, && \text{(A14)}\\
B_0\left(q^2;m^2,m^2\right) &= \frac{2}{\varepsilon} - \gamma_{\scriptscriptstyle E} + \ln\left(\frac{4\pi\mu^2}{m^2}\right) + 2 + v\,\ln\left(\frac{1-v}{1+v}\right), && \text{(A15)}\\
C_0\left(m^2,q^2,m^2;0,m^2,m^2\right) &= \frac{1}{q^2\,v}\Bigg[\left\{\frac{2}{\varepsilon} - \gamma_{\scriptscriptstyle E} + \ln\left(\frac{4\pi\mu^2}{m^2}\right) - \frac{3v^2}{1+v^2}\right\}\ln\left(\frac{1-v}{1+v}\right)\\
&\qquad -\frac{v}{1+v^2}\Big\{F(v)+4\Big\}\Bigg], && \text{(A16)}
\end{aligned}
$$

where $F(v)$ was previously defined in Eq. (16). Here, we have included the 't Hooft mass μ which naturally arises when the couplings are extended to N dimensions.

Thus, the additional real contribution induced by ε in Eq. (A6) is given by

$$
\delta C := C_{\text{bare}} - C = -16\pi i\,\alpha_s C_{\scriptscriptstyle F}\,\varepsilon\,R(v) \;\to\; \frac{\alpha_s}{2\pi}C_{\scriptscriptstyle F} \tag{A17}
$$

Note that the first term in brackets on the right-hand side depends on the mass parameter v and is thus connected with the infrared structure of the theory. On the other hand, the remaining term stems from the ultraviolet sector.

The UV divergences in Eq. (A1) are removed by renormalization. Apart from the conventional on-shell renormalization scheme we take into account the renormalization of the axial-vector current particular to the special γ_5-prescription used [15]. Therefore, the entire renormalization program amounts to isolating C_{rep} in the following equality

$$
1 + C_{\text{rep}} = Z_5\,Z_2\left(1 + C_{\text{bare}}\right), \tag{A18}
$$

where the subscript refers to the particular γ_5-definition chosen. In the $\overline{\text{MS}}$ scheme, the quark-field renormalization constant Z_2 (with gluon mass regularization) and the finite axial-vector renormalization constant Z_5 at one-loop order are given by

$$
\begin{aligned}
Z_2 &= 1 + \frac{\alpha_s}{4\pi}\,C_{\scriptscriptstyle F}\left[-\frac{2}{\varepsilon} + \gamma_{\scriptscriptstyle E} - \ln\left(\frac{4\pi\mu^2}{m^2}\right) - 4 - 2\ln\Lambda + 2\ln\xi - 4\ln 2\right], && \text{(A19)}\\
Z_5 &= 1 - \frac{\alpha_s}{4\pi}\,C_{\scriptscriptstyle F}\Big(4 - 5\,\varepsilon\Big), && \text{(A20)}
\end{aligned}
$$

where the ε in Z_5 generates spurious finite contributions when multiplied with the IR pole terms in the virtual- and real-gluon parts. However, these finite terms cancel in the total results.

In the intermediate step of the derivation, we get after eliminating the UV divergences with Z_2 the relation

$$Z_2\left(1+C_{\text{bare}}\right)=1+C+\frac{\alpha_s}{2\pi}C_F+\delta C, \tag{A21}$$

where C is the chromomagnetic form factor for dimensional regularization with anticommuting γ_5, namely Eq. (14). By multiplying the right-hand side of Eq. (A21) with Z_5, we finally obtain the fully renormalized result (neglecting imaginary parts)

$$\begin{aligned} C_{\text{rep}} &= C+\delta C-\frac{\alpha_s}{2\pi}C_F \\ &= \frac{\alpha_s}{4\pi}C_F\Bigg[2\left(\frac{1+v^2}{v}\ln\left(\frac{1-v}{1+v}\right)-2\right)\left(\ln\Lambda^{\frac{1}{2}}-\tfrac{1}{2}\ln(1-v^2)+\ln 2+1\right) \\ &\qquad -4v\ln\left(\frac{1+v}{1-v}\right)+F(v)+4\Bigg], \end{aligned} \tag{A22}$$

which agrees with the result Eqs. (14) and (23).

Appendix B: Properties of $q\bar{q}g$ Phase-Space Integrals

The massive three-particle phase-space is as usual defined by

$$\mathrm{PS}_3 = \int \left[\prod_{i=1}^{3} \frac{d^N p_i}{(2\pi)^N} \, \delta^+(p_i^2 - m_i^2) \right] (2\pi)^{N+3} \delta \left(q - \sum_{i=1}^{3} p_i \right), \tag{B1}$$

where in this particular case of fermion pair production $m_1 = m_2 = m$. Further, we put $m_3 = \sqrt{q^2}\,\Lambda^{\frac{1}{2}}$ and $N = 4$. Regularizing by a small gluon mass has the advantage to shift all complications to the boundary functions of the integrals whereas dimensional regularization produces an ε dependent integration measure which does in general not permit to fully exploit powerful substitution techniques. It appears that the most complicated of these phase-space integrals are only solvable in four dimensions.

Integrating over the gluon momentum and all the internal angles except for χ gives

$$\mathrm{PS}_3 = \frac{1}{(4\pi)^3} \int \left[\prod_{i=1}^{2} \frac{d|p_i| \, |p_i|^2}{\sqrt{p_i^2 + m^2}} \right] \int_0^{\pi} d\chi \, \sin\chi \; \delta\Big((q - p_1 - p_2)^2 \Big), \tag{B2}$$

where the delta function expresses energy-momentum conservation

$$\begin{aligned} \delta\Big((q - p_1 - p_2)^2\Big) \;&=\; \frac{4}{q^2 \sqrt{(1-y)^2 - \xi}\sqrt{(1-z)^2 - \xi}} \;\times \\ &\qquad \delta\left(\cos\chi - \frac{yz + y + z + \xi - 1 + 2\Lambda}{\sqrt{(1-y)^2 - \xi}\,\sqrt{(1-z)^2 - \xi}} \right). \end{aligned} \tag{B3}$$

Note that we do not put $\Lambda = 0$ in the argument of the delta function which directly effects the deformation of the phase-space boundaries to regulate the IR singularities.

Finally, we obtain with the (y, z)-parametrization given by Eq. (28) for the three-body phase-space

$$\mathrm{PS}_3 = \frac{q^2}{2^7 \pi^3} \int_{y_-}^{y_+} dy \int_{z_-(y)}^{z_+(y)} dz, \tag{B4}$$

where the upper and lower bounds of the nested integral follow directly from the constraint $-1 \leq \cos\chi \leq +1$ and Eq. (B3)

$$\begin{aligned} &\left\{ \begin{array}{rcl} y_+ &=& 1 - \sqrt{\xi}, \\ y_- &=& \Lambda^{\frac{1}{2}}\sqrt{\xi} + \Lambda, \end{array} \right. \\ &z_\pm(y) = \frac{2y}{4y + \xi} \left[1 - y - \tfrac{1}{2}\xi + \Lambda + \frac{\Lambda}{y} \pm \frac{1}{y}\sqrt{(1-y)^2 - \xi}\,\sqrt{(y - \Lambda)^2 - \Lambda\xi} \right]. \end{aligned} \tag{B5}$$

In the calculation of the differential rates one requires the solutions of the following three distinct types of phase-space integrals

$$\mathcal{I}_i = \int dy\,dz\;\; f_i(y,z), \qquad \mathcal{S}_i = \int \frac{dy\,dz}{\sqrt{(1-y)^2-\xi}}\; f_i(y,z),$$
$$\mathcal{J}_i = \int \frac{dy\,dz}{(1-y)^2-\xi}\; f_i(y,z), \tag{B6}$$

where $f_i(y,z)$ are real rational functions in the quark-energy variables $y = 1 - p_1 \cdot q/q^2$ and $z = 1 - p_2 \cdot q/q^2$. Apart from the numerical verification of each individual analytical solution there exist several consistency checks among the different integral classes. For brevity, we will only discuss relations among $\{\mathcal{I}_i\}$ and $\{\mathcal{S}_i\}$, but similar considerations apply to $\{\mathcal{J}_i\}$.

It is important to recognize that although $\{\mathcal{S}_i\}$ and $\{\mathcal{J}_i\}$ emerge as new integrals (in addition to $\{\mathcal{I}_i\}$) in the transition from the the total cross section to the differential form of the cross section, this procedure can not fundamentally alter the divergence structure in the soft divergences [17]. Consider the following argument: The integrand $f(y,z)$ of a divergent $\mathcal{S}$ is first integrated over z to obtain $F(y)$. Then, before executing the remaining y-integration, we add and substract a suitable function $\tilde{F}(y)$:

$$\begin{aligned}\mathcal{S} &= \int \frac{dy\,dz}{\sqrt{(1-y)^2-\xi}}\; f(y,z) = \int \frac{dy}{\sqrt{(1-y)^2-\xi}}\; F(y)\\ &= \int dy\;\tilde{F}(y) + \int \frac{dy}{\sqrt{(1-y)^2-\xi}}\;\left[F(y) - \sqrt{(1-y)^2-\xi}\,\tilde{F}(y)\right].\end{aligned} \tag{B7}$$

Now $\tilde{F}(y)$ can be choosen that way that the second integral on the right-hand side of Eq. (B7) becomes regular as $\Lambda \to 0$ and all soft divergences are contained in the first much simpler integral of type $\mathcal{I}$. It becomes clear that $\mathcal{S}$ and $\mathcal{I}$ exhibit the same divergence structure in Λ, *i.e.* for any $\xi \neq 0$

$$\mathcal{S}(\ln\,) = \mathcal{I}(\ln\,) + \text{finite terms as } \to . \tag{B8}$$

On the other hand, the collinear singularities provide another useful tool to establish a correspondence between $\mathcal{S}$- and $\mathcal{I}$-integrals. For $\Lambda \neq 0$, we can in general find a suitable combination of $\mathcal{S}_i$ and $\mathcal{S}_k$ with $i \neq k$ so that

$$\mathcal{S}_i(\ln\xi, 1/\xi) - \mathcal{S}_i(\ln\xi, 1/\xi) = \mathcal{I}_i(\ln\xi, 1/\xi) + \text{finite terms as } \xi \to 0. \tag{B9}$$

Furthermore, simple algebraic manipulations yield valuable identities that save considerable labor, *e.g.* the following substitution was useful in the derivation of the real-tree graph contributions

$$\int \frac{dy}{(1-y)^2-\xi}\frac{y^2}{z} = \mathcal{I}_2 + 2\mathcal{J}_{12} - (1-\xi)\,\mathcal{J}_8. \tag{B10}$$

An exhaustive list with all three-body phase-space integrals relevant to the derivation of the $\mathcal{O}(\alpha)$ differential cross section for massive fermion pair production is given below. The integral sets $\{\mathcal{I}_i\}$ and $\{\mathcal{S}_i\}$ with the exception of $\mathcal{S}_{13}$ have been published before [7]. The integral class $\{\mathcal{J}_i\}$ is entirely new. Some of its components are also important in the angular dependence of the quark's alignment polarization [19]. For the $\mathcal{J}$-integrals we use the shorthand $w = \sqrt{(1-\sqrt{\xi})/(1+\sqrt{\xi})}$. The standard book on dilogarithms and their integral representations is Ref. [21].

Class $\mathcal{I}$ Integrals

$$\begin{aligned}
\mathcal{I}_1 &= \int dy\,dz \\
&= \tfrac{1}{2}v\left(1+\tfrac{1}{2}\xi\right) - \tfrac{1}{2}\xi\left(1-\tfrac{1}{4}\xi\right)\ln\left(\frac{1+v}{1-v}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_2 &= \int dy\,dz\;\frac{1}{y} \;=\; \int dy\,dz\;\frac{1}{z} \\
&= -v + \left(1-\tfrac{1}{2}\xi\right)\ln\left(\frac{1+v}{1-v}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_3 &= \int dy\,dz\;\frac{1}{y^2} \;=\; \int dy\,dz\;\frac{1}{z^2} \\
&= -\frac{4v}{\xi}\left(\ln\Lambda^{\frac{1}{2}} + \ln\xi - 2\ln v - 2\ln 2 + 1\right) + 2\left(1-\frac{3}{\xi}\right)\ln\left(\frac{1+v}{1-v}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_4 &= \int dy\,dz\;\frac{y}{z} \;=\; \int dy\,dz\;\frac{z}{y} \\
&= -\tfrac{1}{4}v\left(5-\tfrac{1}{2}\xi\right) + \tfrac{1}{2}\left(1+\tfrac{1}{8}\xi^2\right)\ln\left(\frac{1+v}{1-v}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_5 &= \int dy\, dz\ \frac{1}{yz} \\
&= \left(-2\ln\Lambda^{\frac{1}{2}} - \ln\xi + 4\ln v + 2\ln 2\right)\ln\left(\frac{1+v}{1-v}\right) \\
&\quad +2\left[\mathrm{Li}_2\left(\frac{1+v}{2}\right) - \mathrm{Li}_2\left(\frac{1-v}{2}\right)\right] + 3\left[\mathrm{Li}_2\left(-\frac{2v}{1-v}\right) - \mathrm{Li}_2\left(\frac{2v}{1+v}\right)\right]
\end{aligned}$$

Class $\mathcal{S}$ Integrals

$$\begin{aligned}
\mathcal{S}_1 &= \int \frac{dy\, dz}{\sqrt{(1-y)^2-\xi}} \\
&= 1 - \sqrt{\xi} - \tfrac{1}{2}\xi\ln\left(\frac{2-\sqrt{\xi}}{\sqrt{\xi}}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}_2 &= \int \frac{dy\, dz}{\sqrt{(1-y)^2-\xi}}\ \frac{1}{y} \\
&= 2\ln\left(\frac{2-\sqrt{\xi}}{\sqrt{\xi}}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}_3 &= \int \frac{dy\, dz}{\sqrt{(1-y)^2-\xi}}\ \frac{1}{y^2} \\
&= \frac{4}{\xi}\left[-\ln\Lambda^{\frac{1}{2}} + \tfrac{1}{2}\ln\xi + \ln(1-\sqrt{\xi}) - 2\ln(2-\sqrt{\xi}) + \ln 2 - 1\right]
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}_4 &= \int \frac{dy\, dz}{\sqrt{(1-y)^2-\xi}}\ \frac{1}{z} \\
&= \mathrm{Li}_2\left(\frac{1+v}{2}\right) + \mathrm{Li}_2\left(\frac{1-v}{2}\right) + 2\,\mathrm{Li}_2\left(-\frac{\sqrt{\xi}}{2-\sqrt{\xi}}\right) + \tfrac{1}{4}\ln^2\xi \\
&\quad + \ln^2\left(\frac{2-\sqrt{\xi}}{2}\right) - \ln(1+v)\ln(1-v)
\end{aligned}$$

$$\mathcal{S}_5 = \int \frac{dy\, dz}{\sqrt{(1-y)^2-\xi}}\ \frac{1}{z^2}$$

$$= \frac{4}{\xi}\left[-\ln\Lambda^{\frac{1}{2}} - \tfrac{1}{2}\ln\xi + \ln(1-\sqrt{\xi}) - \frac{1+v^2}{2v}\ln\left(\frac{1+v}{1-v}\right) + \ln 2\right]$$

$$\begin{aligned} \mathcal{S}_6 &= \int \frac{dy\,dz}{\sqrt{(1-y)^2-\xi}}\;\frac{y}{z^2} \\ &= \frac{4}{\xi}\left(1-\sqrt{\xi}\right) \end{aligned}$$

$$\begin{aligned} \mathcal{S}_7 &= \int \frac{dy\,dz}{\sqrt{(1-y)^2-\xi}}\;\frac{y^2}{z^2} \\ &= \frac{2}{\xi}\left(1-\sqrt{\xi}\right)^2 \end{aligned}$$

$$\begin{aligned} \mathcal{S}_8 &= \int \frac{dy\,dz}{\sqrt{(1-y)^2-\xi}}\;z \\ &= \frac{1}{32}\left[12-(2+\xi)^2 - \frac{2+\sqrt{\xi}}{2-\sqrt{\xi}}\,\xi^2 + 2(8-\xi)\xi\;\ln\frac{\sqrt{\xi}}{2-\sqrt{\xi}}\right] \end{aligned}$$

$$\begin{aligned} \mathcal{S}_9 &= \int \frac{dy\,dz}{\sqrt{(1-y)^2-\xi}}\;\frac{z}{y} \\ &= -\tfrac{1}{2}\ln\xi + \ln\left(2-\sqrt{\xi}\right) + \frac{2}{2-\sqrt{\xi}} - 2 \end{aligned}$$

$$\begin{aligned} \mathcal{S}_{10} &= \int \frac{dy\,dz}{\sqrt{(1-y)^2-\xi}}\;\frac{y}{z} \\ &= \mathrm{Li}_2\left(\frac{1+v}{2}\right) + \mathrm{Li}_2\left(\frac{1-v}{2}\right) - 2\,\mathrm{Li}_2\left(\frac{\sqrt{\xi}}{2}\right) + \tfrac{1}{4}\ln^2(\tfrac{1}{4}\xi) \\ &\quad + \left(2-\tfrac{1}{2}\xi\right)\ln\left(\frac{2-\sqrt{\xi}}{\sqrt{\xi}}\right) - \sqrt{\xi} + 2v\ln\left(\frac{1-v}{1+v}\right) \\ &\quad - \ln\left(\frac{1+v}{2}\right)\ln\left(\frac{1-v}{2}\right) + 1 \end{aligned}$$

$$
\begin{aligned}
\mathcal{S}_{11} &= \int \frac{dy\,dz}{\sqrt{(1-y)^2-\xi}}\;\frac{y^2}{z} \\
&= \left(1+\tfrac{1}{2}\xi\right)\left[\mathrm{Li}_2\left(\frac{1+v}{2}\right)+\mathrm{Li}_2\left(\frac{1-v}{2}\right)-2\,\mathrm{Li}_2\left(\tfrac{1}{2}\sqrt{\xi}\right)+\tfrac{1}{4}\ln^2\left(\tfrac{1}{4}\xi\right)\right. \\
&\quad \left.-\ln\left(\frac{1+v}{2}\right)\ln\left(\frac{1-v}{2}\right)\right]+3v\ln\left(\frac{1-v}{1+v}\right)+\tfrac{1}{8}(18+\xi)-\tfrac{1}{8}(20-\xi)\sqrt{\xi} \\
&\quad +\left(3-\xi+\tfrac{1}{16}\xi^2\right)\ln\left(\frac{2-\sqrt{\xi}}{\sqrt{\xi}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}_{12} &= \int \frac{dy\,dz}{\sqrt{(1-y)^2-\xi}}\;\frac{1}{yz} \\
&= \frac{1}{v}\ln\left(\frac{1-v}{1+v}\right)\left[2\ln\Lambda^{\frac{1}{2}}+\tfrac{1}{2}\ln\xi+4\ln(2-\sqrt{\xi})-4\ln v-4\ln 2-2\ln\left(\frac{1-v}{1+v}\right)\right] \\
&\quad +\frac{1}{v}\ln^2\left(\frac{(1-v)^2}{\sqrt{\xi}(2-\sqrt{\xi})}\right)+\frac{2}{v}\ln\left(\frac{\sqrt{\xi}(2-\sqrt{\xi})}{2}\right)\ln\left(\frac{2\sqrt{\xi}(1-\sqrt{\xi})}{(1-\sqrt{\xi}-v)^2}\right) \\
&\quad +\frac{2}{v}\left[\mathrm{Li}_2\left(\frac{\sqrt{\xi}(2-\sqrt{\xi})}{(1+v)^2}\right)-\mathrm{Li}_2\left[\left(\frac{1-v}{1+v}\right)^2\right]+\mathrm{Li}_2\left(\frac{(1-v)^2}{\sqrt{\xi}(2-\sqrt{\xi})}\right)\right] \\
&\quad +\frac{1}{v}\left[\mathrm{Li}_2\left(\frac{1+v}{2}\right)-\mathrm{Li}_2\left(\frac{1-v}{2}\right)+\mathrm{Li}_2\left(-\frac{2v}{1-v}\right)-\mathrm{Li}_2\left(\frac{2v}{1+v}\right)-\frac{\pi^2}{3}\right]
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}_{13} &= \int \frac{dy\,dz}{\sqrt{(1-y)^2-\xi}}\;y \\
&= \frac{1}{16}\left[-\xi^2\ln\xi+2\,\xi^2\ln(2-\sqrt{\xi})+4(2-\sqrt{\xi})^2-4\left(2-\xi^{\frac{3}{2}}\right)\right]
\end{aligned}
$$

Class $\mathcal{J}$ Integrals

$$
\begin{aligned}
\mathcal{J}_1 &= \int \frac{dy\,dz}{(1-y)^2-\xi} \\
&= 2\,\frac{1-\xi}{4-\xi}\ln\left(\frac{1+v}{1-v}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_2 &= \int \frac{dy\,dz}{(1-y)^2-\xi}\;\frac{1}{y} \\
&= \frac{6}{4-\xi}\ln\left(\frac{1+v}{1-v}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_3 &= \int \frac{dy\,dz}{(1-y)^2-\xi}\;\frac{1}{y^2} \\
&= \frac{4}{\xi\,v}\Big[-\ln\Lambda^{\frac{1}{2}} - \ln\xi + 2\ln v + 2\ln 2 - 1\Big] - \frac{24}{\xi(4-\xi)}\ln\left(\frac{1+v}{1-v}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_4 &= \int \frac{dy\,dz}{(1-y)^2-\xi}\;y \\
&= -\left(1+\tfrac{1}{2}\xi - \frac{6}{4-\xi}\right)\ln\left(\frac{1+v}{1-v}\right) - v
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_5 &= \int \frac{dy\,dz}{(1-y)^2-\xi}\;y^2 \\
&= \left(2+\tfrac{1}{2}\xi+\tfrac{1}{8}\xi^2 - \frac{6}{4-\xi}\right)\ln\left(\frac{1+v}{1-v}\right) - \tfrac{1}{4}(6-\xi)v
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_6 &= \int \frac{dy\,dz}{(1-y)^2-\xi}\;yz \\
&= \tfrac{1}{2}\xi\left(-2+\tfrac{1}{8}\xi+\frac{9}{4-\xi}-\frac{12}{(4-\xi)^2}\right)\ln\left(\frac{1+v}{1-v}\right) + \left(\tfrac{3}{4}+\tfrac{1}{8}\xi-\frac{2}{4-\xi}\right)v
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_7 &= \int \frac{dy\,dz}{(1-y)^2-\xi}\;z \\
&= -\frac{3\xi}{4-\xi}\left(1-\frac{2}{4-\xi}\right)\ln\left(\frac{1+v}{1-v}\right) + \frac{2v}{4-\xi}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_8 &= \int \frac{dy\,dz}{(1-y)^2-\xi}\;\frac{1}{z} \\
&= \frac{1}{\sqrt{\xi}}\left[\mathrm{Li}_2(w) - \mathrm{Li}_2(-w) + \mathrm{Li}_2\left(\frac{2+\sqrt{\xi}}{2-\sqrt{\xi}}\;w\right) - \mathrm{Li}_2\left(-\frac{2+\sqrt{\xi}}{2-\sqrt{\xi}}\;w\right)\right]
\end{aligned}
$$

$$
\mathcal{J}_9 = \int \frac{dy\,dz}{(1-y)^2-\xi}\;\frac{1}{z^2}
$$

$$= \frac{4}{\xi\, v}\left[-\ln\Lambda^{\frac{1}{2}} - \ln\xi + 2\ln v + 2\ln 2 - \frac{2-\xi}{2v}\ln\left(\frac{1+v}{1-v}\right)\right]$$

$$\begin{aligned}
\mathcal{J}_{10} &= \int \frac{dy\,dz}{(1-y)^2-\xi}\;\frac{1}{yz} \\
&= \frac{1}{2(1-\xi)}\ln\left(\frac{1+v}{1-v}\right)\Big[-4\ln\Lambda^{\frac{1}{2}} - \tfrac{5}{2}\ln\xi + 5\ln(1+\sqrt{\xi}) + 4\ln(1-\sqrt{\xi}) \\
&\qquad - \ln(2+\sqrt{\xi}) + 6\ln 2\Big] \\
&\quad + \frac{2}{1-\xi}\left[\mathrm{Li}_2\left(\frac{1+v}{2}\right) - \mathrm{Li}_2\left(\frac{1-v}{2}\right)\right] + \frac{3}{1-\xi}\left[\mathrm{Li}_2\left(-\frac{2v}{1-v}\right) - \mathrm{Li}_2\left(\frac{2v}{1+v}\right)\right] \\
&\quad + \frac{1}{\sqrt{\xi}(1-\sqrt{\xi})}\left[\mathrm{Li}_2(w) - \mathrm{Li}_2(-w) + \mathrm{Li}_2\left(\frac{2+\sqrt{\xi}}{2-\sqrt{\xi}}\,w\right) - \mathrm{Li}_2\left(-\frac{2+\sqrt{\xi}}{2-\sqrt{\xi}}\,w\right)\right] \\
&\quad - \frac{1}{1-\xi}\Big[\mathrm{Li}_2\left(\frac{1+w}{2}\right) - \mathrm{Li}_2\left(\frac{1-w}{2}\right) + \mathrm{Li}_2\left((2+\sqrt{\xi})\frac{1+w}{4}\right) \\
&\quad - \mathrm{Li}_2\left((2+\sqrt{\xi})\frac{1-w}{4}\right) + \mathrm{Li}_2\left(\frac{2\sqrt{\xi}}{(2+\sqrt{\xi})(1+w)}\right) - \mathrm{Li}_2\left(\frac{2\sqrt{\xi}}{(2+\sqrt{\xi})(1-w)}\right)\Big]
\end{aligned}$$

$$\begin{aligned}
\mathcal{J}_{11} &= \int \frac{dy\,dz}{(1-y)^2-\xi}\;\frac{z}{y} \\
&= \left(-1 + \frac{12}{4-\xi} - \frac{24}{(4-\xi)^2}\right)\ln\left(\frac{1+v}{1-v}\right) - \frac{2v}{4-\xi}
\end{aligned}$$

$$\begin{aligned}
\mathcal{J}_{12} &= \int \frac{dy\,dz}{(1-y)^2-\xi}\;\frac{y}{z} \\
&= \tfrac{1}{2}\ln\left(\frac{1+v}{1-v}\right)\left[\tfrac{1}{2}\ln\xi + \ln(2+\sqrt{\xi}) - \ln(1+\sqrt{\xi}) - 2\ln 2\right] \\
&\quad + \frac{1-\sqrt{\xi}}{\sqrt{\xi}}\left[\mathrm{Li}_2(w) - \mathrm{Li}_2(-w) + \mathrm{Li}_2\left(\frac{2+\sqrt{\xi}}{2-\sqrt{\xi}}\,w\right) - \mathrm{Li}_2\left(-\frac{2+\sqrt{\xi}}{2-\sqrt{\xi}}\,w\right)\right] \\
&\quad + \mathrm{Li}_2\left(\frac{1+w}{2}\right) - \mathrm{Li}_2\left(\frac{1-w}{2}\right) + \mathrm{Li}_2\left((2+\sqrt{\xi})\frac{1+w}{4}\right) - \mathrm{Li}_2\left((2+\sqrt{\xi})\frac{1-w}{4}\right) \\
&\quad + \mathrm{Li}_2\left(\frac{2\sqrt{\xi}}{(2+\sqrt{\xi})(1+w)}\right) - \mathrm{Li}_2\left(\frac{2\sqrt{\xi}}{(2+\sqrt{\xi})(1-w)}\right)
\end{aligned}$$

$$
\begin{aligned}
\mathcal{J}_{13} &= \int \frac{dy\,dz}{(1-y)^2-\xi}\;\frac{y}{z^2} \\
&= \frac{2}{\xi}\ln\left(\frac{1+v}{1-v}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_{14} &= \int \frac{dy\,dz}{(1-y)^2-\xi}\;\frac{y^2}{z^2} \\
&= \frac{2}{\xi}\left[\ln\left(\frac{1+v}{1-v}\right)-2v\right]
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_{15} &= \int \frac{dy\,dz}{(1-y)^2-\xi}\;\frac{y^3}{z} \\
&= \tfrac{1}{2}(3+\xi)\ln\left(\frac{1+v}{1-v}\right)\left[\tfrac{1}{2}\ln\xi+\ln(2+\sqrt{\xi})-\ln(1+\sqrt{\xi})-2\ln 2\right] \\
&\quad +\frac{(1-\sqrt{\xi})^3}{\sqrt{\xi}}\left[\mathrm{Li}_2(w)-\mathrm{Li}_2(-w)+\mathrm{Li}_2\left(\frac{2+\sqrt{\xi}}{2-\sqrt{\xi}}\,w\right)-\mathrm{Li}_2\left(-\frac{2+\sqrt{\xi}}{2-\sqrt{\xi}}\,w\right)\right] \\
&\quad +(3+\xi)\left[\mathrm{Li}_2\left(\frac{1+w}{2}\right)-\mathrm{Li}_2\left(\frac{1-w}{2}\right)+\mathrm{Li}_2\left((2+\sqrt{\xi})\frac{1+w}{4}\right)\right. \\
&\quad \left.-\mathrm{Li}_2\left((2+\sqrt{\xi})\frac{1-w}{4}\right)+\mathrm{Li}_2\left(\frac{2\sqrt{\xi}}{(2+\sqrt{\xi})(1+w)}\right)-\mathrm{Li}_2\left(\frac{2\sqrt{\xi}}{(2+\sqrt{\xi})(1-w)}\right)\right] \\
&\quad +\frac{40-16\xi+\xi^2}{16}\ln\left(\frac{1+v}{1-v}\right)-\frac{26-\xi}{8}\,v
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_{16} &= \int \frac{dy\,dz}{(1-y)^2-\xi}\;\frac{y^3}{z^2} \\
&= \frac{1}{\xi}\left[(2+\xi)\ln\left(\frac{1-v}{1+v}\right)+6v\right]
\end{aligned}
$$

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Figure Captions

Fig. 1: Feynman diagrams contributing to $d\sigma/d\cos\theta$ $(e^+e^- \to q\bar{q})$ up to order $\mathcal{O}(\alpha)$: (a) Born term, (b) virtual corrections, and (c) gluon bremsstrahlung graphs. Diagrams (b) and (c) give the definition of the particle momenta.

Fig. 2: Two-particle kinematics for the process $e^+(p_+)\, e^-(p_-) \;\to\; q(p_1)\, \bar{q}(p_2)$ in the cms coordinate frame. The e^+e^- beam line coincides with the z-axis. The scattering angle between the electron momentum $\boldsymbol{p_-}$ and the quark momentum $\boldsymbol{p_1}$ is θ, and angle φ defines the orientation around the beam axis.

Fig. 3: Three-particle kinematics for the process $e^+(p_+)\, e^-(p_-) \to q(p_1)\, \bar{q}(p_2)\, g(p_3)$ in the cms frame. The momentum definitions agree with those of Fig. 2 except for the additional gluon momentum p_3. The vectors $(\boldsymbol{p_1},\ \boldsymbol{p_2},\ \boldsymbol{p_3})$ span the $q\bar{q}g$ production plane. The angle $\chi \equiv \angle(\boldsymbol{p_1}, \boldsymbol{p_1})$ and the quark momentum $\boldsymbol{p_1}$ uniquely specify all the remaining momenta within the plane.

Fig. 4: Differential cross section for bottom quark production: (a) $\mathcal{O}(\alpha)$ result as a function of the cms energy and $\cos\theta$, and (b) the $\mathcal{O}(\alpha)$ corrections compared to the Born approximation. Shown are the values for the ratio $\Big[d\sigma(\mathcal{O}(\alpha))/d\cos\theta\Big]/\Big[d\sigma(\text{Born})/d\cos\theta\Big]-1$ in percent. The bottom quark mass is $m_b(m_b) = 4.3\,\text{GeV}$.

Fig. 5: $\mathcal{O}(\alpha)$ Differential cross section for top quark production with a top mass of $m_t(m_t) = 174\,\text{GeV}$: (a) surface plot to show functional dependence on the cms energy and scattering angle, and (b) the $\mathcal{O}(\alpha)$ corrections compared to the Born approximation. At 375 GeV the corrections amount to 30 %.